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Asymptotic Estimates for Approximation Quantities of Tensor Product Identities

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Let E_n and F_m be two symmetric Banach spaces, and α a reasonable norm on their tensor product. We give asymptotically best possible estimates for the approximation and Gelfand numbers of the natural embedding from the nm-dimensional Hilbert space l_2^{nm} into $E_n \otimes_\alpha F_m$, and its inverse. Our results are used in order to compute some related characteristics of such tensor products (e.g., type and cotype constants). © 1997 Academic Press

Let $E_n = (\mathbb{R}^n, \|\cdot\|)$ and $F_m = (\mathbb{R}^m, |||\cdot|||)$ be two symmetric Banach spaces and α a reasonable norm on their tensor product $E_n \otimes F_m$. We prove asymptotically best possible estimates for the approximation numbers, Weyl, Gelfand and Kolmogorov numbers of the tensor product identities

$$I_1 = \mathrm{id} \otimes \mathrm{id} \colon l_2^{nm} \to E_n \otimes_\alpha F_m$$

$$I_2 = \mathrm{id} \otimes \mathrm{id} \colon E_n \otimes_\alpha F_m \to l_2^{nm}.$$

We show that the decay of the first $nm/2$ approximation numbers of these identities is very slow: For $i = 1$,2 and all $1 \leqslant k \leqslant [nm/2]$

$$\frac{1}{\sqrt{2}} \|I_i\| \leqslant a_k(I_i) \leqslant \|I_i\|.$$

In several concrete situations the following general conjecture is proved:

$$a_k(I_i) \asymp \max\left(\frac{1}{\|I_i^{-1}\|}, \left(\frac{nm-k+1}{nm}\right)^{1/2} \|I_i\|\right)$$

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(with absolute constants independent of E_n, F_m and α). Using completely different techniques—in particular, the Pajor–Tomczak inequality for Gelfand numbers of operators with values in Hilbert spaces—we show that

$$c_k(I_2)$$

up to a log-term equals the l-norm of the dual of I_2 divided by $(nm)^{1/2}$. For $E_n = l_p^n$, $F_n = l_q^n$ and $\alpha = \varepsilon$ or π (the injective and projective norm), and for the Schatten classes s_p^n our results lead to the precise asymptotic orders of the $[n^2/2]$th approximation, Weyl, Gelfand and Kolmogorov number of id_1 and id_2. Moreover, we prove analogues for Schatten classes of Stechkin's formula for the kth approximation number of $\mathrm{id}: l_1^N \to l_2^N$, and the asymptotic estimate of Garnaev and Gluskin for the kth Gelfand number of $\mathrm{id}: l_1^N \to l_2^N$.

The only article on s-numbers of identity operators on tensor products of l_p^n's we know of is [GKS]; our motivation came from a recent paper of Heinrich [H] which shows that the complexity of computing a functional of a solution of a Fredholm integral equation is related to the asymptotic order of certain tensor product identities. Applications of our results in this direction will be given in a forthcoming paper; in the present paper our estimates are used to prove asymptotically best possible bounds for some (local Banach space) invariants of finite dimensional tensor products—e.g. the type 2 constant of $l_p^n \otimes_\pi l_q^n$ and cotype 2 constant of $l_p^n \otimes_\varepsilon l_q^n$.

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We always consider real Banach spaces X and denote their unit ball by B_X. For a linear and continuous operator $T \in \mathscr{L}(X, Y)$ (between Banach spaces) recall the definition of the kth approximation number

$$a_k(T) := \inf\{ \|T - R\| \mid R \in \mathscr{L}(X, Y), \text{rank } R < k\},$$

the kth Weyl number

$$x_k(T) := \sup\{a_k(TR) \mid R \in \mathscr{L}(l_2, X), \|R\| \leqslant 1\},$$

the kth Gelfand number

$$c_k(T) := \inf\{ \|T_{|G}\| \mid G \subset X, \text{codim } G < k\}$$

and the kth Kolmogorov number

$$d_k(T) := \inf\{ \|q_L T\| \mid L \subset Y, \dim L < k\},$$

where q_L denotes the quotient mapping $E \to E/L$. For $s = a, x, c, d$ the sequences $(s_k(T))$ are non-increasing, $s_1(T) = \|T\|$, $s_n(\mathrm{id}_{l_2^n}) = 1$, and $s_k(T) = 0$ whenever rank $T < k$. It is known that $x_k \leqslant c_k \leqslant a_k$ (hence equality for operators on Hilbert spaces) and $d_k \leqslant a_k$; if T is compact, then $c_k(T) = d_k(T')$ and $d_k(T) = c_k(T')$. Moreover, $c_k(T) = a_k(I_Y T)$ and $d_k(T) = a_k(T Q_X)$ where $I_Y \colon Y \hookrightarrow l_\infty(B_{Y'})$ and $Q_X \colon l_1(B_X) \to X$ denote the canonical mappings. Finally, we recall that all these s-number scales are multiplicative, i.e.

$$s_{k+n-1}(ST) \leqslant s_k(S)\, s_n(T) \qquad \text{for appropriate } S, T.$$

For more information see [CS], [K], [P2], [P], and [Pi].

For two Banach spaces E and F we write $E \otimes_\pi F$ for the projective tensor product, and $E \otimes_\varepsilon F$ for the injective tensor product. Moreover, for $1 \leqslant p \leqslant \infty$ we denote by $l_p \otimes_{\Delta p} E$ the space $l_p \otimes E$ endowed with the norm coming from the inclusion $l_p \otimes E \hookrightarrow l_p(E)$; recall that $\varepsilon \leqslant \Delta_p \leqslant \pi$. The space $\mathscr{L}(l_2^n, l_2^n)$ together with the Schatten p-norm is denoted by s_p^n; it is well-known that $s_1^n = l_2^n \otimes_\pi l_2^n$, $s_2^n = l_2^n \otimes_{\Delta_2} l_2^n = l_2^{n^2}$ and $s_\infty^n = l_2^n \otimes_\varepsilon l_2^n$. We use [DF] as a general reference for tensor products of Banach spaces.

1

A well-known result of Pietsch [P2], 2.9.8 states that for $1 \leqslant p < q \leqslant \infty$ and $1 \leqslant k \leqslant N$

$$a_k(\mathrm{id} \colon l_q^N \to l_p^N) = (N - k + 1)^{1/p - 1/q};$$

in particular, for $1 \leqslant k \leqslant [N/2]$

$$\frac{1}{\sqrt{2}}\, \|\mathrm{id}\| \leqslant a_k(\mathrm{id}) \leqslant \|\mathrm{id}\|$$

—the first $[N/2]$-approximation numbers almost equal the norm (here $[N/2]$ stands for the *smallest* integer larger than or equal to $N/2$). The Gelfand and Kolmogorov number satisfy the same formula.

For the special case $q = 2$ and $p = 1$ we have the following extension.

Proposition. *For $m, n \in \mathbb{N}$ let α be a norm on $l_1^n \otimes l_1^m$ with $\varepsilon \leqslant \alpha \leqslant \pi$. Then for all $1 \leqslant k \leqslant nm$*

$$a_k(\mathrm{id} \colon l_2^{nm} \to l_1^n \otimes_\alpha l_1^m) = (nm - k + 1)^{1/2};$$

in particular, for $1 \leqslant k \leqslant [nm/2]$

$$\frac{1}{\sqrt{2}}\,\|\mathrm{id}\| \leqslant a_k(\mathrm{id}) \leqslant \|\mathrm{id}\|.$$

The proof is based on a simple lemma. Recall that for $T \in \mathscr{L}(X, Y)$ the absolutely p-summing norm $(1 \leqslant p < \infty)$ is given by

$$\pi_p(T) := \sup\left\{\left(\sum_{k=1}^{n} \|Tx_k\|^p\right)^{1/p} \,\middle|\, \sup_{B_{E'}}\left(\sum_{k=1}^{n} |x'(x_k)|^p\right)^{1/p} \leqslant 1\right\} \in [0, \infty].$$

For operators between Hilberts spaces this ideal norm coincides with the Hilbert Schmidt norm **HS**(= Schatten 2-norm), and

$$\pi_2(\mathrm{id}_X) = \sqrt{N} \qquad \text{whenever} \quad \dim X = N;$$

see e.g. [DF], [P1] or [T] for details.

Lemma. *Let* $T \in \mathscr{L}(X, Y)$ *be an invertible operator between two N-dimensional Banach spaces X and Y. Then for all* $1 \leqslant k \leqslant N$

$$c_k(T) \geqslant \frac{(N-k+1)^{1/2}}{\pi_2(T^{-1})}.$$

Proof. Take a subspace $M \subset X$ with codim $M < k$. Then

$$N - k + 1 \leqslant \dim M,$$

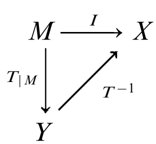
hence

$$(N-k+1)^{1/2} \leqslant (\dim M)^{1/2} = \pi_2(\mathrm{id}_M).$$

Clearly (by the injectivity of π_2)

$$\pi_2(\mathrm{id}_M) = \pi_2(I\colon M \hookrightarrow X),$$

therefore,



gives, as desired,

$$(N-k+1)^{1/2} \leqslant \|T_{|M}\|\, \pi_2(T^{-1}). \quad \blacksquare$$

The proof of the proposition now follows easily: Since $l_1^n \otimes_\pi l_1^m = l_1^{nm}$, the result for $\alpha = \pi$ obviously is a special case of Pietsch's formula. So it is enough to check the lower bound for $\alpha = \varepsilon$. It is well-known (see e.g. [FJ]) that π_2 is tensor stable in the following sense: For $T \in \mathcal{L}(E, l_2^n)$ and $S \in \mathcal{L}(F, l_2^m)$

$$\pi_2(T \otimes S \colon E \otimes_\varepsilon F \to l_2^{nm}) = \pi_2(T)\, \pi_2(S).$$

Since (see e.g. [P1])

$$\pi_2(\mathrm{id} \colon l_1^N \hookrightarrow l_2^N) = 1,$$

the lemma gives

$$\begin{aligned} a_k&(\mathrm{id} \colon l_2^{nm} \to l_1^n \otimes_\varepsilon l_1^m) \\ &\geqslant (nm - k + 1)^{1/2}\, \pi_2(\mathrm{id} \colon l_1^n \otimes_\varepsilon l_1^m \to l_2^{nm})^{-1} \\ &= (nm - k + 1)^{1/2}\, \pi_2(\mathrm{id} \colon l_1^n \hookrightarrow l_2^n)^{-1}\, \pi_2(\mathrm{id} \colon l_1^m \hookrightarrow l_2^m)^{-1} \\ &= (nm - k + 1)^{1/2}. \end{aligned}$$

This completes the proof. ∎

Clearly, the proposition also holds for the Gelfand and Weyl numbers—but it will be seen in section 6 that it does not hold for the Kolmogorov numbers (and $\alpha = \varepsilon$).

2

The second statement of the proposition can be improved considerably which needs some preparation.

For $n \in \mathbb{N}$ denote by Π_n the set of all permutations of $\{1, ..., n\}$ and by $\mathcal{D}_n$ the set of all $(\varepsilon_k)_{k=1}^n$ with $\varepsilon_k = \pm 1$. For $\varepsilon \in \mathcal{D}_n$ and $\pi \in \Pi_n$ let

$$D_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n, \qquad D_\varepsilon x := \sum_{k=1}^n \varepsilon_k x_k e_k$$

$$P_\pi \colon \mathbb{R}^n \to \mathbb{R}^n, \qquad P_\pi x := \sum_{k=1}^n x_{\pi(k)} e_k.$$

If $\|\cdot\|$ is some norm on $\mathbb{R}^n$, then $X = (\mathbb{R}^n, \|\cdot\|)$ is said to be symmetric whenever all D_ε and P_π define isometries on X. It is easy to check that with X also X' has this property. The most important examples are the l_p^n's or $\mathbb{R}^n$ with some Orlicz norm. We call a norm α on the tensor product

$E_n \otimes F_m$ of two such spaces symmetrically invariant if $\varepsilon \leqslant \alpha \leqslant \pi$ and for all symmetries $S \in \mathscr{S}_n := \{D_\varepsilon \mid \varepsilon \in \mathscr{D}_n\} \cup \{P_\pi \mid \in \Pi_n\}$ and $T \in \mathscr{S}_m$

$$S \otimes T \colon E_n \otimes_\alpha F_m \to E_n \otimes_\alpha F_m$$

is an isometry. All tensor norms—in particular, ε and π—are symmetrically invariant, and also Δ_p and the Schatten p-norm have this property.

The following result is one of our main tools—it seems to be known to some specialists. Therefore we only sketch the proof.

PROPOSITION. *Let α and β be symmetrically invariant norms on $E_n \otimes F_m$ and $X_n \otimes Y_m$, respectively, where all spaces are symmetric,* $\dim E_n = \dim X_n = n$ *and* $\dim F_m = \dim Y_m = m$. *Then*

$$\pi_2(\mathrm{id} \colon E_n \otimes_\alpha F_m \to X_n \otimes_\beta Y_m) = (nm)^{1/2} \frac{\|\mathrm{id} \colon l_2^{nm} \to X_n \otimes_\beta Y_m\|}{\|\mathrm{id} \colon l_2^{nm} \to E_n \otimes_\alpha F_m\|}. \tag{1}$$

Clearly, (1) *has as special cases*

$$\pi_2(\mathrm{id} \colon E_n \otimes_\alpha F_m \to l_2^{nm}) = (nm)^{1/2} \|\mathrm{id} \colon l_2^{nm} \to E_n \otimes_\alpha F_m\|^{-1} \tag{2}$$

and

$$\pi_2(\mathrm{id} \colon l_2^{nm} \to X_n \otimes_\beta Y_m) = (nm)^{1/2} \|\mathrm{id} \colon l_2^{nm} \to X_n \otimes_\beta Y_m\|. \tag{3}$$

For unitarily invariant norms α on tensor products of Hilbert spaces equality (2)—at least essentially—seems to be due to [GL], [L], and is explicitly stated in [T], p. 310; our proof is completely elementary and modelled along similar lines.

Assume for a moment that the upper estimate in (2) has been proven. Then (1) can be derived by standard arguments as follows: The upper estimate is a consequence of

$$\begin{aligned} &\pi_2(\mathrm{id} \colon E_n \otimes_\alpha F_m \to X_n \otimes_\beta Y_m) \\ &\qquad \leqslant \pi_2(\mathrm{id} \colon E_n \otimes_\alpha F_m \to l_2^{nm}) \, \|\mathrm{id} \colon l_2^{nm} \to X_n \otimes_\beta Y_m\|, \end{aligned}$$

and the lower estimate is obtained by trace duality (see e.g. [DF], p. 208, 232, or [P1]) since

$$\begin{aligned} nm &\leqslant \pi_2(\mathrm{id} \colon l_2^{nm} \to X_n \otimes_\beta Y_m) \, \pi_2(\mathrm{id} \colon X_n \otimes_\beta Y_m \to l_2^{nm}) \\ &\leqslant \|\mathrm{id} \colon l_2^{nm} \to E_n \otimes_\alpha F_m\| \\ &\qquad \times \pi_2(\mathrm{id} \colon E_n \otimes_\alpha F_m \to X_n \otimes_\beta Y_m) \, \pi_2(\mathrm{id} \colon X_n \otimes_\beta Y_m \to l_2^{nm}). \end{aligned}$$

For the proof of the upper estimate in (2) we prefer to change the setting—the following statement is a reformulation of (2) in terms of linear operators:

(2′) *For E_n and F_m as above let* $\mathbf{A}$ *be a symmetrically invariant norm on* $\mathcal{L}(E_n, F_m)$, *i.e. for all symmetries* $S \in \mathcal{S}_n$ *and* $T \in \mathcal{S}_m$

$$\mathbf{A}(TUS) = \mathbf{A}(U) \qquad \textit{for all} \quad U \in \mathcal{L}(E_n, F_m).$$

Then

$$\pi_2(\mathrm{id}: (\mathcal{L}(E_n, F_m), \mathbf{A}) \to (\mathcal{L}(l_2^n, l_2^m), \mathbf{HS}))$$
$$= (nm)^{1/2} \, \|\mathrm{id}: (\mathcal{L}(l_2^n, l_2^m), \mathbf{HS}) \to (\mathcal{L}(E_n, F_m), \mathbf{A})\|^{-1}.$$

In order to see that (2) is an immediate consequence of (2′) apply (2′) to the symmetrically invariant norm $\mathbf{A}$ defined by

$$(\mathcal{L}(E_n', F_m), \mathbf{A}) := E_n \otimes_\alpha F_m$$

(recall that with E_n also E_n' is symmetric).

For the proof of (2′) a non-commutative version of the Khinchine *equality* for Rademacher 2-averages is needed. Let ν_n be the Haar measure on Π_n, i.e.

$$\nu_n(\{\pi\}) := \frac{1}{n!} \qquad \text{for all} \quad \pi \in \Pi_n,$$

and μ_n the Haar measure on $\mathcal{D}_n$ given by

$$\mu_n(\{\varepsilon\}) := \frac{1}{2^n} \qquad \text{for all} \quad \varepsilon \in \mathcal{D}_n.$$

Lemma 1. *For* $R \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *and* $S \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$

$$\left(\int_{\Pi_n} \int_{\mathcal{D}_n} \int_{\Pi_m} \int_{\mathcal{D}_m} |tr(RD_\varepsilon P_\pi S D_{\tilde\varepsilon} P_{\tilde\pi})|^2 \, d\mu_m(\tilde\varepsilon) \, d\nu_m(\tilde\pi) \, d\mu_n(\varepsilon) \, d\nu_n(\pi) \right)^{1/2}$$
$$= \frac{\mathbf{HS}(R) \, \mathbf{HS}(S)}{(nm)^{1/2}}. \tag{4}$$

In order to picture this formula look at

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{R} & \mathbb{R}^n \\ {\scriptstyle D_\varepsilon P_\pi}\uparrow & & \downarrow{\scriptstyle D_{\tilde\varepsilon} P_{\tilde\pi}} \\ \mathbb{R}^n & \xleftarrow[S]{} & \mathbb{R}^m. \end{array}$$

For its proof an elementary lemma helps.

LEMMA 2. *For any* $x, y \in \mathbb{R}^n$

$$\left(\int_{\Pi_n}\int_{\mathscr{D}_n} |\langle x, D_\varepsilon P_\pi y\rangle|^2\, d\mu_n(\varepsilon)\, d\nu_n(\pi)\right)^{1/2} = \frac{\|x\|_2\, \|y\|_2}{n^{1/2}}.$$

Proof of Lemma 2 (for abbreviation we write $d\varepsilon := d\mu_n(\varepsilon)$ and $d\pi := d\nu_n(\pi)$). Without loss of generality we show the formula for $y = e_1$:

$$\begin{aligned}\int_{\Pi_n}\int_{\mathscr{D}_n} |\langle x, \varepsilon_{\pi(1)} e_{\pi(1)}\rangle|^2\, d\varepsilon\, d\pi &= \int_{\Pi_n} |\langle x, e_{\pi(1)}\rangle|^2\, d\pi\\ &= \sum_{l=1}^{n} \int_{\pi(1)=l} |x_{\pi(1)}|^2\, d\pi\\ &= \sum_{l=1}^{n} \frac{1}{n!}(n-1)!\, |x_l|^2\\ &= \frac{1}{n}\|x\|_2^2. \quad \blacksquare\end{aligned}$$

The formula (4) of Lemma 1 follows immediately from Lemma 2 and the definitions of the trace and **HS**-norm of operators T by

$$tr(T) = \sum_i \langle Te_i, e_i\rangle$$

and

$$\mathbf{HS}(T) = \left(\sum_i \|Te_i\|^2\right)^{1/2} = \left(\sum_i \|T^* e_i\|^2\right)^{1/2}$$

(see also [T], p. 310). $\blacksquare$

The proof of (2′) now more or less repeats the elementary part of Pietsch's domination theorem [P1], p. 232. Namely, let S be an element of the unit ball B of the Banach space $(\mathscr{L}(E_n, E_m), \mathbf{A})'$ such that

$$\mathbf{HS}(S) = \sup\{\mathbf{HS}(T) \mid T \in B\},$$

and μ the image of the counting measure $d\varepsilon\, d\tilde{\varepsilon}\, d\pi\, d\tilde{\pi}$ on the set

$$\{D_\varepsilon P_\pi S D_{\tilde{\varepsilon}} P_{\tilde{\pi}}\} \subset B.$$

Then by Lemma 1 for any $T \in \mathcal{L}(E_n, F_m)$ one has

$$\mathbf{HS}(T) = \frac{(nm)^{1/2}}{\mathbf{HS}(S)} \left(\int_B |tr(TR)|^2 \, d\mu(R) \right)^{1/2}$$

Now the conclusion follows as in Pietsch's theorem.

3

For $T \in \mathcal{L}(E, F)$ and $k \in \mathbb{N}$

$$k^{1/2} x_k(T) \leqslant \pi_2(T)$$

(see e.g. [K] and [P1]). This is the crucial link between Weyl/approximation numbers and the 2-summing norm which together with the proposition of the preceding section now easily gives the following estimate.

Proposition. *Let* α *and* β *be symmetrically invariant norms on* $E_n \otimes F_m$ *and* $X_n \otimes Y_m$, *respectively, where all spaces are symmetric*, $\dim E_n = \dim X_n = n$ *and* $\dim F_m = \dim Y_m = m$. *Then for all* $1 \leqslant k \leqslant nm$

$$\begin{aligned} \left(\frac{nm-k+1}{nm} \right)^{1/2} & \frac{\|\mathrm{id} : l_2^{nm} \to X_n \otimes_\beta Y_m\|}{\|\mathrm{id} : l_2^{nm} \to E_n \otimes_\alpha F_m\|} \\ & \leqslant x_k(\mathrm{id} : E_n \otimes_\alpha F_m \to X_n \otimes_\beta Y_m) \\ & \leqslant \left(\frac{nm}{k} \right)^{1/2} \frac{\|\mathrm{id} : l_2^{nm} \to X_n \otimes_\beta Y_m\|}{\|\mathrm{id} : l_2^{nm} \to E_n \otimes_\alpha F_m\|}. \end{aligned}$$

Proof. The second inequality is obvious from what was said before, and the first then follows from the basic properties of the Weyl numbers:

$$\begin{aligned} 1 &= x_{nm}(\mathrm{id}_{l_2^{nm}}) \\ &\leqslant x_k(\mathrm{id} : l_2^{nm} \to X_n \otimes_\beta Y_m) \, x_{nm-k+1}(\mathrm{id} : X_n \otimes_\beta Y_m \to l_2^{nm}) \\ &\leqslant \|\mathrm{id} : l_2^{nm} \to E_n \otimes_\alpha F_m\| \, x_k(\mathrm{id} : E_n \otimes_\alpha F_m \to X_n \otimes_\beta Y_m) \\ &\quad \times \left(\frac{nm}{nm-k+1} \right)^{1/2} \|\mathrm{id} : l_2^{nm} \to X_n \otimes_\beta Y_m\|^{-1}. \quad \blacksquare \end{aligned}$$

There are immediate consequences of this result.

COROLLARY. *Let $E_n \otimes_\alpha F_m$ and $X_n \otimes_\beta Y_m$ be as above.*

(1) *For* $1 \leqslant k \leqslant [nm/2]$

$$\frac{1}{\sqrt{2}} \|\mathrm{id}\| \leqslant a_k(\mathrm{id}\colon l_2^{nm} \to X_n \otimes_\beta Y_m)$$

$$= c_k(\mathrm{id}\colon l_2^{nm} \to X_n \otimes_\beta Y_m) \leqslant \|\mathrm{id}\|.$$

(2) *For* $1 \leqslant k \leqslant [nm/2]$

$$\frac{1}{\sqrt{2}} \|\mathrm{id}\| \leqslant a_k(\mathrm{id}\colon E_n \otimes_\alpha F_m \to l_2^{nm})$$

$$= d_k(\mathrm{id}\colon E_n \otimes_\alpha F_m \to l_2^{nm}) \leqslant \|\mathrm{id}\|.$$

(3) $$\frac{1}{\sqrt{2}} \frac{\|\mathrm{id}\colon l_2^{nm} \to X_n \otimes_\beta Y_m\|}{\|\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m\|} \leqslant x_{[nm/2]}(\mathrm{id}\colon E_n \otimes_\alpha F_m \to X_n \otimes_\beta Y_m)$$

$$\leqslant \sqrt{2} \frac{\|\mathrm{id}\colon l_2^{nm} \to X_n \otimes_\beta Y_m\|}{\|\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m\|}.$$

Let us now interpret these results for the special spaces $E_n = l_p^n$, $F_m = l_q^m$ and the norms $\alpha = \varepsilon$ or π; define

$$\alpha(n, m, p, q) := \|\mathrm{id}\colon l_2^{nm} \to l_p^n \otimes_\alpha l_q^m\|.$$

We know by the mapping property for ε (see [DF], p. 46) that

$$\varepsilon(n, m, p, q) = \|\mathrm{id}\colon l_2^n \to l_p^n\| \, \|\mathrm{id}\colon l_2^m \to l_q^m\|$$

$$= \begin{cases} n^{1/p-1/2} m^{1/q-1/2} & 1 \leqslant p, \quad q \leqslant 2 \\ 1 & 2 \leqslant p, \quad q \leqslant \infty \\ n^{1/p-1/2} & 1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty \\ m^{1/q-1/2} & 1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty. \end{cases}$$

Such asymptotic estimates for π are more involved: For $n, m \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$

$$\pi(n, m, p, p) \asymp \begin{cases} 1 & 4 \leqslant p \leqslant \infty \\ \min(n, m)^{2/p-1/2} & 2 \leqslant p \leqslant 4 \\ (nm)^{1/2} \max(n, m)^{1/p-1} & 1 \leqslant p \leqslant 2, \end{cases}$$

and for $n \in \mathbb{N}$, $1 \leqslant p \leqslant q \leqslant \infty$

$$\pi(n, n, p, q) \asymp \begin{cases} 1 & p \geqslant 2, \quad 1/p + 1/q \leqslant 1/2 \\ n^{1/p + 1/q - 1/2} & p \geqslant 2, \quad 1/p + 1/q \geqslant 1/2 \\ n^{1/q} & 1 \leqslant p \leqslant q \leqslant 2 \\ n^{1/2} & 1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty, \quad p \leqslant q' \\ n^{1/p + 1/q - 1/2} & 1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty, \quad q' \leqslant p. \end{cases}$$

Note that the constants depend only on p and q, and that the asymptotic order for $1 \leqslant q \leqslant p \leqslant \infty$ clearly follows by symmetry. Some of these estimates go back to Hardy and Littlewood [HL]—the whole collections can be found in [S1], [S2]. Clearly, estimates for $\|\mathrm{id}\colon l_p^n \otimes_\alpha l_q^m \to l_2^{nm}\|$ can be obtained by the well-known duality of ε and π (see e.g. [DF], Section 6).

In particular, we get for $\alpha, \beta \in \{\varepsilon, \pi\}$ and $p, q, r, s \in [1, \infty]$ the optimal asymptotic growth (in terms of p and q) of

$$a_{[n^2/2]}(\mathrm{id}\colon l_2^{n^2} \to l_p^n \otimes_\alpha l_q^n) = c_{[n^2/2]}(\mathrm{id})$$

$$a_{[n^2/2]}(\mathrm{id}\colon l_p^n \otimes_\alpha l_q^n \to l_2^{n^2}) = d_{[n^2/2]}(\mathrm{id})$$

$$x_{[n^2/2]}(\mathrm{id}\colon l_p^n \otimes_\alpha l_q^n \to l_r^n \otimes_\beta l_s^n).$$

For Schatten p-classes the corollary gives

$$a_{[n^2/2]}(\mathrm{id}\colon s_2^n \to s_p^n) = c_{[n^2/2]}(\mathrm{id}) \asymp \max(1, n^{1/p - 1/2})$$

$$a_{[n^2/2]}(\mathrm{id}\colon s_p^n \to s_2^n) = d_{[n^2/2]}(\mathrm{id}) \asymp \max(1, n^{1/2 - 1/p})$$

$$x_{[n^2/2]}(\mathrm{id}\colon s_p^n \to s_q^n) \asymp \frac{\max(1, n^{1/q - 1/2})}{\max(1, n^{1/p - 1/2})}.$$

The proposition can also be used to complete some of the estimates from [GKS]. For example in [GKS], 2.9 for $1 < p < 2$ the asymptotic order of

$$a_k(\mathrm{id}\colon l_p^n \otimes_\pi l_p^n \to l_{p'}^n \otimes_\varepsilon l_{p'}^n)$$

is calculated—with one gap: For $[n^2/2] \leqslant k \leqslant n^2 - [n^{2/p'}]$ only the upper estimate

$$a_k(\mathrm{id}) \leqslant d_p \, \frac{(n^2 - k + 1)^{1/2}}{n^{1 + 1/p}}$$

is given. The proposition yields that this bound is optimal:

$$a_k(\mathrm{id}) \geqslant x_k(\mathrm{id}) \geqslant \left(\frac{n^2-k+1}{n^2}\right)^{1/2} \frac{\|\mathrm{id}: l_2^{n^2} \to l_{p'}^n \otimes_\varepsilon l_{p'}^n\|}{\|\mathrm{id}: l_2^{n^2} \to l_p^n \otimes_\pi l_p^n\|}$$
$$= \frac{(n^2-k+1)^{1/2}}{n} \frac{1}{n^{1/p}}.$$

We close this section with the following estimate related to a conjecture of Heinrich [H].

Remark. *Let $1 \leqslant p \leqslant 2$. Then for E_n, F_m and α as in the proposition*

$$2^{-1/2}(nm)^{1/2-1/p} \|\mathrm{id}: l_2^{nm} \to E_n \otimes_\alpha F_m\| \leqslant c_{[nm/2]}(\mathrm{id}: l_p^{nm} \to E_n \otimes_\alpha F_m)$$
$$\leqslant d(nm)^{1/2-1/p} \|\mathrm{id}: l_2^{nm} \to E_n \otimes_\alpha F_m\|,$$

where $d > 0$ is universal.

Proof. The first inequality follows from the corollary by factoring the identity $\mathrm{id}: l_2^{nm} \to E_n \otimes_\alpha F_m$ through l_p^{nm}, and the second one from the fact that

$$c_{[nm/2]}(\mathrm{id}: l_p^{nm} \to l_2^{nm}) \prec (nm)^{1/2-1/p}$$

(this is a consequence of the Pajor–Tomczak inequality which we will recall in section 4). ∎

For $2 \leqslant p \leqslant \infty$ it seems to be reasonable to conjecture that there is a universal constant $d > 0$ such that for all n, m

$$d \|\mathrm{id}: l_p^{nm} \to E_n \otimes_\alpha F_m\| \leqslant c_{[nm/2]}(\mathrm{id}) \leqslant \|\mathrm{id}\|;$$

for the special case $p = \infty$, $\alpha = \varepsilon$ and $E_n = F_n = l_1^n$ this would answer a problem of Heinrich [H].

4

In section 6 we will deal with the cases

$$c_{[n^2/2]}(\mathrm{id}: l_p^n \otimes_\alpha l_q^n \to l_2^{n^2})$$
$$d_{[n^2/2]}(\mathrm{id}: l_2^{n^2} \to l_p^n \otimes_\alpha l_q^n),$$

for which completely different techniques are needed. The theory of Gelfand numbers for operators with values in a Hilbert space is ruled by the following deep inequality of Pajor and Tomczak-Jaegermann [PT]: There is a universal constant $c>0$ such that for all $T \in \mathscr{L}(X, l_2^N)$ and $1 \leqslant k \leqslant N$

$$k^{1/2} c_k(T) \leqslant c l(T').$$

Recall that the l-norm for $S \in \mathscr{L}(l_2^N, Y)$ is given by

$$l(S) := \left(\int_{\mathbb{R}^N} \left\| \sum_{k=1}^{N} g_k(\omega)\, Te_k \right\|^2 \gamma_N(d\omega) \right)^{1/2},$$

where γ_N is the N-dimensional Gauss measure on $\mathbb{R}^N$ and $g_k \colon \mathbb{R}^N \to \mathbb{R}$ the kth projection.

Proposition. *There are universal constants $c, d > 0$ such that for each pair of symmetric Banach spaces E_n and F_m, E_n n-dimensional and F_m m-dimensional and every symmetrically invariant norm α on $E_n \otimes F_m$*

$$c_{[nm/2]}(\mathrm{id} \colon E_n \otimes_\alpha F_m \to l_2^{nm}) \leqslant c\, \frac{l(\mathrm{id} \colon l_2^{nm} \to E'_n \otimes_{\alpha'} F'_m)}{(nm)^{1/2}} \tag{1}$$

and up to a logarithmic term this result is asymptotically best possible:

$$\frac{1}{d}\, \frac{l(\mathrm{id} \colon l_2^{nm} \to E'_n \otimes_{\alpha'} F'_m)}{(1+\log nm)(nm)^{1/2}} \leqslant c_{[nm/2]}(\mathrm{id} \colon E_n \otimes_\alpha F_m \to l_2^{nm}), \tag{2}$$

here α' is the dual norm of α defined by $E'_n \otimes_{\alpha'} F'_m := (E_n \otimes_\alpha F_m)'$.

Clearly only (2) needs a proof. For this denote the ellipsoid of maximal volume contained in the unit ball $B_{E_n \otimes_\alpha F_m}$ of $E_n \otimes_\alpha F_m$ by $D_{\max}$ (see [P] or [T] for this notion).

Lemma 1. *For E_n, F_m and α as above*

$$D_{\max} = \| \mathrm{id} \colon l_2^{nm} \to E_n \otimes_\alpha F_m \|^{-1} B_{l_2^{nm}}.$$

Proof. Consider $U := \|\mathrm{id}\|^{-1}\, \mathrm{id}$. Then by the proposition of section 2

$$\pi_2(U) = \pi_2(U^{-1}) = (nm)^{1/2}.$$

On the other hand for any linear bijection generating $D_{\max}$:

$$V \colon l_2^{nm} \to E_n \otimes_\alpha F_m \qquad \text{with} \qquad V(B_{l_2^{nm}}) = D_{\max},$$

we also have

$$\pi_2(V) = \pi_2(V^{-1}) = (nm)^{1/2}.$$

Hence Lewis' uniqueness theorem implies that $U^{-1}V$ is an isometry (for these two well-known results on $D_{\max}$ see e.g. [P], 3.8 and 3.6). ▮

$E_n \otimes_\alpha F_m$ has enough symmetries (for this notion see [T], and for a proof of this fact [GL]). Hence, if $\|\cdot\|_{\max}$ denotes the euclidean norm generated by $D_{\max}$ and

$$I\colon (\mathbb{R}^{nm}, \|\cdot\|_{\max}) \to E_n \otimes_\alpha F_m$$

stands for the identity, then by a result of [BG] on Banach–Mazur distances d (between spaces with enough symmetries and Hilbert spaces, see also [T], p. 131)

$$d(E_n \otimes_\alpha F_m, l_2^{nm}) = \|I\| \, \|I^{-1}\|.$$

By the corollary this implies a result of Schütt [S1]—a fact which will be needed later:

$$d(E_n \otimes_\alpha F_m, l_2^{nm}) = \|\mathrm{id}\| \, \|\mathrm{id}^{-1}\|.$$

Using trace duality and the reformulation $d(X, l_2^n) = \mathbf{L}_2(\mathrm{id}_X)$, the $\mathbf{L}_2$-factorable norm of id_X (see e.g. [DF] or [P1]), it is also possible to deduce this directly from statement (2) of the proposition in Section 2.

LEMMA 2. *For* $\mathrm{id}\colon E_n \otimes_\alpha F_m \to l_2^{nm}$

$$nm \leqslant l(\mathrm{id}^{-1}) \, l(\mathrm{id}') \leqslant \gamma(1 + \log nm) \, nm,$$

here E_n, F_m *and* α *are again as above and* $\gamma > 0$ *is some universal constant.*

Proof. Since $E_n \otimes_\alpha F_m$ has enough symmetries, it follows from a result of [BG] (see also [T], p. 131) that

$$nm = l(I) \, l^*(I^{-1}).$$

Moreover, for some universal $\gamma > 0$

$$l^*(I^{-1}) \leqslant l((I^{-1})') \leqslant \gamma(1 + \log nm) \, l^*(I^{-1})$$

([T], p. 87, 92), hence finally

$$\begin{aligned} nm &\leqslant l(I) \, l((I^{-1})') = l(\mathrm{id}^{-1}) \, l(\mathrm{id}') \\ &\leqslant \gamma(1 + \log nm) \, l(I) \, l^*(I^{-1}) = \gamma(1 + \log nm) \, nm. \end{aligned}$$ ▮

Lemma 3. *Let E and F be two N-dimensional Banach spaces. Then for each invertible* $S \in \mathscr{L}(E, F)$ *and* $1 \leqslant k \leqslant N$

$$\frac{1}{c_{N-k+1}((S^{-1})')} \leqslant c_k(S).$$

Proof. We will need the following numbers which for $T \in \mathscr{L}(X, Y)$ and $k \in \mathbb{N}$ are defined by

$$t_k(T) := a_k(I_Y T Q_X).$$

These numbers were first introduced and studied by Ismagilov [I] under the name of absolute width (cf. Tichomirov numbers in [P1] or symmetrized approximation numbers in [CS]). By Tichomirov's theorem we have

$$t_n(\mathrm{id}_X) = 1 \qquad \text{whenever} \quad \dim X = n$$

(cf. [Pi]). Hence the conclusion follows from the multiplicativity of the approximation numbers, and the fact that the Gelfand and Kolmogorov numbers are dual to each other:

$$\begin{aligned} 1 &= t_N(SS^{-1}) = a_N(I_F S S^{-1} Q_F) \\ &\leqslant a_k(I_F S)\, a_{N-k+1}(S^{-1} Q_F) \\ &= c_k(S)\, d_{N-k+1}(S^{-1}) = c_k(S)\, c_{N-k+1}((S^{-1})'). \quad \blacksquare \end{aligned}$$

We now easily obtain a proof of Part (2) of the proposition:

$$\begin{aligned} c_{[nm/2]}(\mathrm{id}) &\geqslant \frac{1}{c_{[nm/2]}((\mathrm{id}^{-1})')} \geqslant \frac{1}{c} \frac{(nm)^{1/2}}{l(\mathrm{id}^{-1})} \\ &\geqslant \frac{1}{c\gamma} \frac{(nm)^{1/2}\, l(\mathrm{id}')}{nm(1+\log nm)} = \frac{1}{d} \frac{l(\mathrm{id}')}{(nm)^{1/2}\,(1+\log nm)}. \quad \blacksquare \end{aligned}$$

In general the log-term in (2) is not superfluous—to see this recall a celebrated result of Garnaev and Gluskin [GG] (see also [P], p. 81): For $1 \leqslant k \leqslant N$

$$c_k(\mathrm{id}\colon l_1^N \to l_2^N) \asymp \min\left(1, \left(\frac{\log(1+N/k)}{k}\right)^{1/2}\right).$$

Since $l(\mathrm{id}\colon l_1^N \to l_\infty^N) \asymp (1+\log N)^{1/2}$ (see the next section), this shows that for $\alpha = \pi$, $E_n = l_1^n$ and $F_m = l_1^m$ the denominator of the left side of (2) at least needs the term $(1+\log nm)^{1/2}$.

5

As an application and for later use we calculate the asymptotic order of the (Gaussian) cotype 2 and (Gaussian) type 2 constant of $l_p^n \otimes_\varepsilon l_q^n$ and $l_p^n \otimes_\pi l_q^n$, respectively. Recall that a Banach space E has cotype 2 if there is a constant $c \geqslant 0$ such that for all $x_1, ..., x_n \in E$

$$\left(\sum_{k=1}^{n} \|x_k\|^2 \right)^{1/2} \leqslant c \left(\int_{\mathbb{R}^n} \left\| \sum_{k=1}^{n} g_k x_k \right\|^2 d\gamma_n \right)^{1/2},$$

and type 2 if

$$\left(\int_{\mathbb{R}^n} \left\| \sum_{k=1}^{n} g_k x_k \right\|^2 d\gamma_n \right)^{1/2} \leqslant c \left(\sum_{k=1}^{n} \|x_k\|^2 \right)^{1/2}.$$

Moreover, $\mathbf{C}_2(E) := \inf c$ and $\mathbf{T}_2(E) := \inf c$ are called cotype 2 and type 2 constant of E, respectively. It is well-known (see e.g. [T], p. 15) that

$$\mathbf{C}_2(l_p^n) \asymp \begin{cases} 1 & 1 \leqslant p \leqslant 2 \\ n^{1/2 - 1/p} & 2 \leqslant p < \infty \\ \dfrac{n^{1/2}}{(1 + \log n)^{1/2}} & p = \infty \end{cases}$$

$$\mathbf{T}_2(l_q^n) \asymp \begin{cases} n^{1/q - 1/2} & 1 \leqslant q \leqslant 2 \\ 1 & 2 \leqslant q < \infty \\ (1 + \log n)^{1/2} & q = \infty. \end{cases}$$

There is a useful observation (see [P], p. 151) relating approximation numbers, cotype 2 constants and l-norms: For any $T \in \mathscr{L}(l_2^N, E)$ and all $1 \leqslant k \leqslant N$

$$k^{1/2} a_k(T) \leqslant \mathbf{C}_2(E)\, l(T).$$

For the estimation of the l-norms of the tensor product identities under consideration we moreover need Chevet's inequality on Gaussian averages which has the following useful reformulation in terms of l-norms and ε-tensor products (see [T], p. 318): There is a constant $c > 0$ such that for all $S \in \mathscr{L}(l_2^n, E)$ and $T \in \mathscr{L}(l_2^m, F)$

$$\max(\|S\|\, l(T), l(S)\, \|T\|) \leqslant l(S \otimes T \colon l_2^{nm} \to E \otimes_\varepsilon F)$$
$$\leqslant c(\|S\|\, l(T) + l(S)\, \|T\|).$$

Since

$$l(\mathrm{id}\colon l_2^N \to l_p^N) \asymp \begin{cases} N^{1/p} & 1 \leqslant p < \infty \\ (1+\log N)^{1/2} & p = \infty \end{cases}$$

([T], p. 329), one easily derives the following asymptotic estimates.

Remark. *For* $1 \leqslant p \leqslant q \leqslant \infty$

$$l(\mathrm{id}\colon l_2^{n^2} \to l_p^n \otimes_\varepsilon l_q^n) \asymp \begin{cases} n^{1/p+1/q-1/2} & 1 \leqslant q \leqslant 2 \\ n^{1/p} & 2 \leqslant q \leqslant \infty, \quad p < \infty \\ (1+\log n)^{1/2} & p = q = \infty. \end{cases}$$

Now everything is prepard for the proof of the following application.

PROPOSITION (1) *For* $1 \leqslant p \leqslant q \leqslant \infty$, $(p, q) \neq (\infty, \infty)$

$$\mathbf{C}_2(l_p^n \otimes_\varepsilon l_q^n) \asymp n^{1/2} \mathbf{C}_2(l_p^n) \asymp \begin{cases} n^{1/2} & p \leqslant 2 \\ n^{1-1/p} & p \geqslant 2. \end{cases}$$

(2) *For* $1 \leqslant p \leqslant q < \infty$

$$\mathbf{T}_2(l_p^n \otimes_\pi l_q^n) \asymp n^{1/2} \mathbf{T}_2(l_q^n) \asymp \begin{cases} n^{1/q} & q \leqslant 2 \\ n^{1/2} & q \geqslant 2. \end{cases}$$

For the remaining case $1 \leqslant p \leqslant \infty$, $q = \infty$ *we have*:

$$\mathbf{T}_2(l_p^n \otimes_\pi l_\infty^n) \asymp n^{1/2} \qquad \textit{for} \quad 2 \leqslant p < \infty$$

$$n^{1/2} \prec \mathbf{T}_2(l_p^n \otimes_\pi l_\infty^n) \prec n^{1/2}(1+\log n)^{1/2} \qquad \textit{for} \quad 1 < p < 2 \quad \textit{or} \quad p = \infty$$

$$\mathbf{T}_2(l_1^2 \otimes_\pi l_\infty^n) \asymp n^{1/2}(1+\log n)^{1/2}.$$

We don't know whether the logarithmic term in the second statement is superfluous.

Proof. The lower estimate in (1) is a consequence of

$$[n^2/2]^{1/2}\, a_{[n^2/2]}(\mathrm{id}\colon l_2^{n^2} \to l_p^n \otimes_\varepsilon l_q^n) \leqslant \mathbf{C}_2(l_p^n \otimes_\varepsilon l_q^n)\, l(\mathrm{id}),$$

the estimate for the approximation numbers from section 3 and the preceding remark. Next we prove the upper estimate in (2): Recall that for any operator $T \in \mathscr{L}(E, F)$, $1 \leqslant p < \infty$ and $n \in \mathbb{N}$

$$\begin{aligned}\pi_p(T) &= \|\mathrm{id}\otimes T\colon l_p\otimes_\varepsilon E\to l_p\otimes_{\Delta_p} F\|\\ &= \|\mathrm{id}\otimes T'\colon l_{p'}\otimes_{\Delta_{p'}} F'\to l_{p'}\otimes_\pi E'\|\\ &\geqslant \|\mathrm{id}\otimes T\colon l_p^n\otimes_\varepsilon E\to l_p^n\otimes_{\Delta_p} E\|\\ &= \|\mathrm{id}\otimes T'\colon l_{p'}^n\otimes_{\Delta_{p'}} F'\to l_{p'}^n\otimes_\pi E'\|\end{aligned}$$

([DF], p. 127), and for finite dimensional E

$$\begin{aligned}&\mathbf{C}_2(l_p^n(E))\leqslant c\,\mathbf{C}_2(E), && 1\leqslant p\leqslant 2\\ &\mathbf{T}_2(l_q^n(E))\leqslant c\,\mathbf{T}_2(E), && 2\leqslant q<\infty\end{aligned}$$

($c>0$ universal, [T], p. 17). Hence we obtain for $2\leqslant p\leqslant q<\infty$

$$\begin{aligned}\mathbf{T}_2(l_q^2\otimes_\pi l_p^n) &\leqslant \|l_q^n\otimes_\pi l_p^n\xrightarrow{\mathrm{id}} l_q^n\otimes_{\Delta_q} l_p^n\|\,\mathbf{T}_2(l_q^n(l_p^n))\\ &\quad\times\|l_q^n\otimes_{\Delta_q} l_p^n\xrightarrow{\mathrm{id}} l_q^n\otimes_\pi l_p^n\|\\ &\prec \mathbf{T}_2(l_p^n)\,\pi_{q'}(\mathrm{id}_{l_{p'}^n})\prec n^{1/2}\end{aligned}$$

([P1], p. 312), for $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$

$$\begin{aligned}\mathbf{T}_2(l_p^n\otimes_\pi l_q^n) &\leqslant \|l_p^n\otimes_\pi l_q^n\xrightarrow{\mathrm{id}} l_2^n\otimes_{\Delta_2} l_q^n\|\,\mathbf{T}_2(l_2^n(l_q^n))\\ &\quad\times\|l_2^n\otimes_{\Delta_2} l_q^n\xrightarrow{\mathrm{id}} l_p^n\otimes_\pi l_q^n\|\\ &\prec \mathbf{T}_2(l_q^n)\,\|l_2^n\otimes_{\Delta_2} l_q^n\xrightarrow{\mathrm{id}} l_p^n\otimes_\pi l_q^n\|\\ &\leqslant \mathbf{T}_2(l_q^n)\,\|l_2^n\otimes_{\Delta_2} l_q^n\xrightarrow{\mathrm{id}} l_1^n\otimes_\pi l_q^n\|\\ &\leqslant n^{1/2}\cdot\begin{cases}1 & q<\infty\\ (1+\log n)^{1/2} & q=\infty\end{cases}\end{aligned}$$

(Hölder's inequality, see also [DF], 7.3), for $1\leqslant p\leqslant q\leqslant 2$

$$\begin{aligned}\mathbf{T}_2(l_p^n\otimes_\pi l_q^n) &\leqslant \|l_p^n\otimes_\pi l_q^n\xrightarrow{\mathrm{id}} l_2^n\otimes_{\Delta_2} l_2^n\|\\ &\quad\times\|l_2^n\otimes_{\Delta_2} l_2^n\xrightarrow{\mathrm{id}} l_p^n\otimes_\pi l_q^n\|\\ &\prec n^{1/q}\end{aligned}$$

(section 3), and finally for $2\leqslant p\leqslant\infty$

$$\begin{aligned}\mathbf{T}_2(l_\infty^n\otimes_\pi l_p^n) &\leqslant \|l_\infty^n\otimes_\pi l_p^n\xrightarrow{\mathrm{id}} l_2^n\otimes_{\Delta_2} l_p^n\|\,\mathbf{T}_2(l_2^n(l_p^n))\\ &\quad\times\|l_2^n\otimes_{\Delta_2} l_p^n\xrightarrow{\mathrm{id}} l_\infty^n\otimes_\pi l_p^n\|\\ &\prec n^{1/2}\mathbf{T}_2(l_p^n)\,\|l_2^n\otimes_{\Delta_p} l_p^n\xrightarrow{\mathrm{id}} l_\infty^n\otimes_\pi l_p^n\|\end{aligned}$$

$$\leqslant n^{1/2}\mathbf{T}_2(l_p^n)\,\pi_{p'}(l_1^n \xrightarrow{\text{id}} l_2^n)$$
$$\leqslant n^{1/2}\mathbf{T}_2(l_p^n)\,\pi_1(l_1^n \xrightarrow{\text{id}} l_2^n)$$
$$\prec n^{1/2}\cdot\begin{cases}1 & p<\infty\\ (1+\log n)^{1/2} & p=\infty\end{cases}$$

(Hölder's inequality, [DF], 7.2 and [P1], p. 312). Since for arbitrary r, s

$$\mathbf{C}_2(l_r^n\otimes_\varepsilon l_s^n)\leqslant \mathbf{T}_2(l_{r'}^n\otimes_\pi l_{s'}^n)$$

(see e.g. [T] or [DF], p. 106), this ends the proof of (2)—with one exception: the proof of the lower estimate of the last statement in (2) is postponed to the remarks after the proposition in the next section. In order to prove the upper estimate in (1) note that for $1\leqslant q\leqslant 2$

$$\begin{aligned}\mathbf{C}_2(l_1^n\otimes_\varepsilon l_q^n)&\leqslant \|l_1^n\otimes_\varepsilon l_q^n \xrightarrow{\text{id}} l_1^n\otimes_\pi l_q^n\|\;\mathbf{C}_2(l_1^n(l_q^n))\\ &\quad\times\|l_1^n\otimes_\pi l_q^n\xrightarrow{\text{id}} l_1^n\otimes_\varepsilon l_q^n\|\\ &\prec \mathbf{C}_2(l_q^n)\,\pi_1(\mathrm{id}_{l_q^n})\prec n^{1/2}\end{aligned}$$

([P1], p. 312), and for $2\leqslant q\leqslant\infty$

$$\begin{aligned}\mathbf{C}_2(l_1^n\otimes_\varepsilon l_q^n)&\leqslant \|l_1^n\otimes_\varepsilon l_\infty^n \xrightarrow{\text{id}} l_1^n\otimes_{\Delta_2} l_2^n\|\;\mathbf{C}_2(l_2^n(l_1^n))\\ &\quad\times\|l_1^n\otimes_{\Delta_2} l_2^n\xrightarrow{\text{id}} l_1^n\otimes_\varepsilon l_q^n\|\prec n^{1/2}\end{aligned}$$

(Hölder's inequality), hence the above duality argument also finishes the proof of (1). ∎

6

As announced at the beginning of section 4 we now complete the results from section 3.

Proposition. *For* $1\leqslant p\leqslant q\leqslant\infty$

$$c_{[n^2/2]}(\mathrm{id}\colon l_p^n\otimes_\varepsilon l_q^n\to l_2^{n^2})\asymp\begin{cases}n^{1-1/p} & q\geqslant 2\\ n^{3/2-1/p-1/q} & q\leqslant 2\end{cases}\tag{1}$$

$$c_{[n^2/2]}(\mathrm{id}\colon l_p^n\otimes_\pi l_q^n\to l_2^{n^2})\asymp\begin{cases}n^{1/2-1/p-1/q} & p\geqslant 2\\ n^{-1/q} & p\leqslant 2,\end{cases}\tag{2}$$

and by duality one obtains a corresponding result for Kolmogorov numbers. Moreover, as a by-product we get for $1 \leqslant p \leqslant q < \infty$

$$l(\mathrm{id}\colon l_2^{n^2} \to l_p^n \otimes_\pi l_q^n) \asymp \begin{cases} n^{1/2+1/p+1/q} & p \geqslant 2 \\ n^{1+1/q} & p \leqslant 2. \end{cases} \tag{3}$$

The constants in (1), (2) and (3) depend on p and q only. For asymptotic estimates for the Schatten p-norms see (4) at the end of this section.

Proof. We start with the upper estimate for (3) which is based on the fact that for $S \in \mathscr{L}(l_2^N, Y)$

$$l(S) \leqslant \mathbf{T}_2(Y)\, \pi_2(S')$$

(see e.g. [T], p. 83). Together with the results from the proposition in section 2, the asymptotic order of $\varepsilon(n, n, p', q')$ given in section 3 and the proposition from section 5 this yields the upper bound in (3). With this in hand the Pajor–Tomczak inequality gives the upper estimate in (1) whenever $1 < p \leqslant q \leqslant \infty$. The remaining cases can be obtained as follows: For $1 \leqslant q \leqslant 2$

$$\begin{aligned} &c_{[n^2/2]}(\mathrm{id}\colon l_1^n \otimes_\varepsilon l_q^n \to l_2^{n^2}) \\ &\quad \leqslant n^{1-1/q}\, \|\mathrm{id}\colon l_1^n \otimes_\varepsilon l_1^n \to l_1^n \otimes_\pi l_1^n\|\, c_{[n^2/2]}(\mathrm{id}\colon l_1^{n^2} \to l_2^{n^2}) \\ &\quad = n^{1-1/q} \pi_1(\mathrm{id}_{l_1^n})\, n^{-1} \prec n^{1-1/q} n^{1/2} n^{-1} \\ &\quad = n^{1/2-1/q} \end{aligned}$$

and for $2 \leqslant q \leqslant \infty$

$$\begin{aligned} c_{[n^2/2]}(\mathrm{id}\colon l_1^n \otimes_\varepsilon l_q^n \to l_2^{n^2}) &\leqslant \|\mathrm{id}\colon l_1^n \otimes_\varepsilon l_q^n \to l_1^n \otimes_\pi l_1^n\|\, c_{[n^2/2]}(\mathrm{id}\colon l_1^{n^2} \to l_2^{n^2}) \\ &= \pi_1(\mathrm{id}\colon l_q^n \to l_1^n)\, n^{-1} \prec n n^{-1} = 1 \end{aligned}$$

(use [P1], p. 312 and the Garnaev–Gluskin estimate mentioned at the end of section 4). Analogously, the upper bound in (2) is a consequence of the remark in section 5 and the Pajor–Tomczak inequality provided that $(p, q) \neq (1, 1)$; for $p = q = 1$ see again the end of section 4. This finishes the proofs of (1) and (2) since Lemma 3 from section 4 combined with the upper estimates yields the lower ones. Finally, the missing lower estimate in (3) follows from the lower estimate in (1) and another application of the Pajor–Tomczak inequality. ∎

Using the same ideas we obtain in (3) for the remaining case $1 \leqslant p \leqslant \infty$, $q = \infty$:

$$l(\mathrm{id}\colon l_2^{n^2} \to l_p^n \otimes_\pi l_\infty^n) \asymp n^{1/2+1/p} \qquad \text{for} \quad 2 \leqslant p < \infty$$
$$n \prec l(\mathrm{id}) \prec (1+\log n)^{1/2}\, n \qquad \text{for} \quad 1 \leqslant p < 2 \quad \text{or} \quad p = \infty$$
$$n(1+\log n)^{1/2} \asymp l(\mathrm{id}\colon l_2^{n^2} \to l_1^n \otimes_\pi l_\infty^n),$$

and again we don't know whether the log-term in the second statement is superfluous; for the lower bound in the third statement note that for $1 \leqslant p < \infty$

$$n^{1/p}(1+\log m)^{1/2} \asymp l(\mathrm{id}\colon l_2^{nm} \to l_p^n(l_\infty^m))$$

which follows by direct calculation using the fact that $l(\mathrm{id}\colon l_2^m \to l_\infty^m) \asymp (1+\log m)^{1/2}$. This now allows to prove the lower estimate of the last statement of the proposition in section 5:

$$\begin{aligned} n(1+\log n)^{1/2} &\asymp l(\mathrm{id}\colon l_2^{n^2} \to l_1^n \otimes_\pi l_\infty^n) \\ &\leqslant \mathbf{T}_2(l_1^n \otimes_\pi l_\infty^n)\, \pi_2(\mathrm{id}') \\ &= \mathbf{T}_2(l_1^n \otimes_\pi l_\infty^n)\, \frac{n}{\|(\mathrm{id}')^{-1}\|}, \end{aligned}$$

hence $n^{1/2}(1+\log n)^{1/2} \prec \mathbf{T}_2(l_1^n \otimes_\pi l_\infty^n)$.

For Schatten p-classes the methods yield the following asymptotic order:

$$c_{[n^2/2]}(\mathrm{id}\colon s_p^n \to s_2^n) \asymp n^{1/2-1/p}, \qquad 1 \leqslant p \leqslant \infty. \tag{4}$$

Again the upper bound is a consequence of the Pajor–Tomczak inequality combined with

$$l(\mathrm{id}\colon s_2^n \to s_r^n) \prec n^{1/2+1/r}, \qquad 1 \leqslant r \leqslant \infty$$

(see e.g. [T], p. 329), and the lower estimate then follows from Lemma 3 in section 4.

7

Let α be a symmetrically invariant norm on the tensor product of two symmetric Banach spaces E_n and F_m, E_n n-dimensional and F_m

m-dimensional. Recall from the proposition in section 3 that the first $[nm/2]$ approximation numbers of

$$\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m$$

equal the operator norm of id (up to the constant $1/\sqrt{2}$). In this section we collect some estimates for the indices $[nm/2] \leqslant k \leqslant nm$.

Note first that by the proposition in section 3

$$\left(\frac{nm-k+1}{nm}\right)^{1/2} \|\mathrm{id}\| \leqslant x_k(\mathrm{id}) = a_k(\mathrm{id}),$$

and trivially

$$1 = a_{nm}(\mathrm{id}_{l_2^{nm}}) \leqslant a_k(\mathrm{id})\ \|\mathrm{id}^{-1}\|,$$

which proves that for all $1 \leqslant k \leqslant nm$

$$\max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \left(\frac{nm-k+1}{nm}\right)^{1/2} \|\mathrm{id}\|\right) \leqslant a_k(\mathrm{id}) \leqslant \|\mathrm{id}\|.$$

For $1 \leqslant k \leqslant [nm/2]$ the left side (up to a constant) equals $\|\mathrm{id}\|$ since we obviously have that $\|\mathrm{id}\|\ \|\mathrm{id}^{-1}\| \geqslant 1$.

Conjecture. There is a universal constant $c > 0$ such that for all E_n, F_m and α as above and all $[nm/2] \leqslant k \leqslant nm$

$$a_k(\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m) \leqslant c \max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \left(\frac{nm-k+1}{nm}\right)^{1/2} \|\mathrm{id}\|\right).$$

The following remarks show why this conjecture seems to be reasonable.

A. The remark after Lemma 1 of section 4 proves

$$\|\mathrm{id}\|\ \|\mathrm{id}^{-1}\| \leqslant d(l_2^{nm}, E_n \otimes_\alpha F_m) \leqslant (nm)^{1/2}$$

(for the latter estimate see e.g. [T]), hence we obtain from [CD], p. 72

$$a_{nm}(\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m) = \frac{1}{x_1(\mathrm{id}^{-1})}$$

$$= \max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \left(\frac{nm-nm+1}{nm}\right)^{1/2} \|\mathrm{id}\|\right).$$

B. By the proposition from section 1 for all $1 \leqslant k \leqslant nm$

$$a_k(\mathrm{id}\colon l_2^{nm} \to l_1^n \otimes_\alpha l_1^m) \asymp \max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \left(\frac{nm-k+1}{nm}\right)^{1/2} \|\mathrm{id}\|\right),$$

since $\|\mathrm{id}\| \asymp (nm)^{1/2}$ and $\|\mathrm{id}^{-1}\| \asymp 1$ (see section 3).

C. The same holds for $\alpha = \varepsilon$, $E_n = l_\infty^n$ and $F_m = l_\infty^m$ because of a well-known result of Stechkin (see e.g. [P2]):

$$a_k(\mathrm{id}\colon l_1^N \to l_2^N) = a_k(\mathrm{id}\colon l_2^N \to l_\infty^N) = \left(\frac{N-k+1}{N}\right)^{1/2}.$$

D. Steckin's result can be extended with the help of the following inequality based on a probabilistic estimate from [GKS]:

LEMMA. *For all E_n, F_m and α as above and $[nm/2] \leqslant k \leqslant nm$*

$$a_k(\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m) \leqslant c \max\left(\frac{l(\mathrm{id})}{(nm)^{1/2}}, \left(\frac{nm-k+1}{nm}\right)^{1/2} \|\mathrm{id}\|\right),$$

where c is an absolute constant.

Proof. We know from [GKS], 2.2 that

$$a_k(\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m) \leqslant \frac{l(\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m) + (nm-k+1)^{1/2} \|\mathrm{id}\colon l_2^{nm} \to E_n \otimes_\alpha F_m\|}{l(\mathrm{id}\colon l_2^{nm} \to l_2^{nm}) - (nm-k+1)^{1/2} \|\mathrm{id}\colon l_2^{nm} \to l_2^{nm}\|}$$

Since $l(\mathrm{id}\colon l_2^{nm} \to l_2^{nm}) = (nm)^{1/2}$ and $[nm/2] \leqslant k \leqslant nm$, this implies the desired result. ∎

REMARK 1. *For $1 \leqslant k \leqslant n^2$*

$$a_k(\mathrm{id}\colon l_2^{n^2} \to l_p^n \otimes_\varepsilon l_q^n) \asymp \max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \left(\frac{n^2-k+1}{n}\right)^{1/2} \|\mathrm{id}\|\right),$$

whenever p and q satisfy one of the following three cases:

$$2 \leqslant p \leqslant q \leqslant \infty$$

$$1 \leqslant p \leqslant q \leqslant 2 \quad \textit{and} \quad 1/p + 1/q \leqslant 3/2$$

$$p = q = 1$$

(*with constants only depending on p, q*).

Proof. **A** and **C** cover the cases $(p, q) = (1, 1)$ and $(p, q) = (\infty, \infty)$. For all other cases the remark of section 5 and the estimates for $\pi(n, n, p', q')$ of section 3 give

$$\frac{l(\mathrm{id})}{n} \asymp \frac{1}{\|\mathrm{id}^{-1}\|}. \quad \blacksquare$$

REMARK 2. *For $1 \leqslant k \leqslant n^2$ and $2 \leqslant p \leqslant \infty$*

$$a_k(\mathrm{id}: s_2^n \to s_p^n) \asymp \max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \left(\frac{n^2 - k + 1}{n}\right)^{1/2} \|\mathrm{id}\|\right);$$

for $p = \infty$ this is an analogue of Stechkin's result from **C** *for Schatten classes*:

$$a_k(\mathrm{id}: s_1^n \to s_2^n) = a_k(\mathrm{id}: s_2^n \to s_\infty^n)$$
$$\asymp \begin{cases} 1 & 1 \leqslant k \leqslant [n^2/2] \\ \dfrac{(n^2 - k + 1)^{1/2}}{n} & [n^2/2] \leqslant k \leqslant n^2 - n + 1 \\ \dfrac{1}{n^{1/2}} & n^2 - n + 1 \leqslant k \leqslant n^2. \end{cases}$$

Proof. Everything follows from what was said before, the lemma and the fact (see [T], p. 329) that

$$l(\mathrm{id}: s_2^n \to s_p^n) \prec n^{1/2 + 1/p} = n \frac{1}{\|\mathrm{id}^{-1}\|}. \quad \blacksquare$$

E. By duality all results mentioned so far can be formulated for

$$a_k(\mathrm{id}: E_n \otimes_\alpha F_m \to l_2^{nm}).$$

F. Let us now turn to the asymptotic growth of

$$c_k(\mathrm{id}: E_n \otimes_\alpha F_m \to l_2^{nm}).$$

Direct comparison of the estimates from sections 3 and 6 shows that for $1 \leqslant k \leqslant n^2/2$

$$c_k(\mathrm{id}: l_p^n \otimes_\varepsilon l_q^n \to l_2^{n^2}) \asymp \|\mathrm{id}\|,$$

whenever $2 \leqslant p \leqslant q \leqslant \infty$ or $1 \leqslant p \leqslant q \leqslant 2$, $1/p + 1/q \leqslant 3/2$. Moreover, we have

$$c_k(\mathrm{id}: s_p^n \to s_2^{n^2}) \asymp \|\mathrm{id}\|,$$

for $1 \leqslant k \leqslant n^2/2$ and $2 \leqslant p \leqslant \infty$. Using what was said in the proofs of **D**, Remark 1 and 2, this can also be seen as a consequence of the following general result.

LEMMA. *Let E_n, F_m and α be as above. Then for all $1 \leqslant k \leqslant nm$*

$$c_k(\mathrm{id}\colon E_n \otimes_\alpha F_m \to l_2^{nm}) \geqslant \max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \frac{1}{c}\frac{(nm-k+1)^{1/2}}{l(\mathrm{id}^{-1})}\right),$$

where $c > 0$ is the constant from the Pajor–Tomczak inequality.

The proof is an immediate consequence of the Pajor–Tomczak inequality combined with Lemma 3, section 4.

We finish this section with the following analogue of the Garnaev–Gluskin result (mentioned at the end of section 4) for Schatten classes:

REMARK. *For $1 \leqslant p \leqslant 2$ and $1 \leqslant k \leqslant n^2$*

$$c_k(\mathrm{id}\colon s_p^n \to s_2^n) \asymp \min\left(1, \frac{l(\mathrm{id}')}{k^{1/2}}\right)$$
$$= \begin{cases} 1 & 1 \leqslant k \leqslant [n^{3-2/p}] \\ \dfrac{n^{3/2-1/p}}{k^{1/2}} & [n^{3-2/p}] \leqslant k \leqslant [n^2/2] \\ n^{1/2-1/p} & [n^2/2] \leqslant k \leqslant n^2 \end{cases}$$

(*the constants only depend on p*). *In particular*,

$$c_k(\mathrm{id}\colon s_1^n \to s_2^n) \asymp \begin{cases} 1 & 1 \leqslant k \leqslant n \\ \dfrac{n^{1/2}}{k^{1/2}} & n \leqslant k \leqslant [n^2/2] \\ \dfrac{1}{n^{1/2}} & [n^2/2] \leqslant k \leqslant n^2. \end{cases}$$

Proof. By The Pajor–Tomczak inequality we get

$$c_k(\mathrm{id}) \leqslant c \min\left(\|\mathrm{id}\|, \frac{l(\mathrm{id}')}{k^{1/2}}\right)$$

which together with $l(\mathrm{id}') \prec n^{1/p'+1/2}$ gives the upper estimate. For the lower estimate note first that for $[n^2/2] \leqslant k \leqslant n^2$

$$c_k(\mathrm{id}) \geqslant c_{n^2}(\mathrm{id}) \geqslant \frac{1}{\|\mathrm{id}^{-1}\|} = n^{1/2-1/p}.$$

Since for $1 \leqslant k \leqslant [n^{3-2/p}]$

$$c_{[n^{3-2/p}]}(\mathrm{id}) \leqslant c_k(\mathrm{id}),$$

it suffices to show that for $[n^{3-2/p}] \leqslant k \leqslant [n^2/2]$

$$c_k(\mathrm{id}) \succ \frac{n^{3/2-1/p}}{k^{1/2}}.$$

This can be seen by use of a result from [GKS], 2.3: For $[n^{3-2/p}] \leqslant k \leqslant [n^2/2]$

$$\begin{aligned} c_k(\mathrm{id}) &= d_k(\mathrm{id}' : s_2^n \to s_{p'}^n) \\ &\geqslant \frac{n^2 - \sqrt{kn^2}}{\sqrt{kn^2}\, n^{1/2-1/p'} + n^2} \\ &\geqslant \frac{n^{3/2-1/p}}{k^{1/2}} \frac{1-1/\sqrt{2}}{2}. \qquad \blacksquare \end{aligned}$$

8

We finally give asymptotically best possible bounds for the $[nm/2]$th entropy number of

$$\mathrm{id}: E_n \otimes_\alpha F_m \to l_2^{nm},$$

and compare these results with Schütt's volume estimates for the unit balls of tensor products of l_r^n's (see [S2]).

Recall that the kth entropy number of $T \in \mathscr{L}(X, Y)$ is defined by

$$e_k(T) := \inf \left\{ \varepsilon > 0 \mid \exists y_1, ..., y_{2^{k-1}} \in Y: TB_X \subset \bigcup_{l=1}^{2^{k-1}} y_l + \varepsilon B_Y \right\}$$

(see [CS] or [P]). By a result of Milman and Pisier for all $T \in \mathscr{L}(X, l_2^N)$

$$c_{[N/2]}(T) \leqslant c e_{[N/2]}(T),$$

and by Sudakov's inequality for all such T and $1 \leqslant k \leqslant N$

$$k^{1/2} e_k(T) \leqslant d l(T')$$

(here $c, d \geqslant 0$ are universal constants, see [P], p. 68, 81). Hence we conclude from (and as in) (1), (2) and (4) of section 6:

PROPOSITION. *For* $1 \leqslant p \leqslant q \leqslant \infty$

$$e_{[n^2/2]}(\mathrm{id}\colon l_p^n \otimes_\varepsilon l_q^n \to l_2^{n^2}) \asymp c_{[n^2/2]}(\mathrm{id}) \asymp \begin{cases} n^{1-1/p} & q \geqslant 2 \\ n^{3/2-1/p-1/q} & q \leqslant 2 \end{cases} \tag{1}$$

$$e_{[n^2/2]}(\mathrm{id}\colon l_p^n \otimes_\pi l_q^n \to l_2^{n^2}) \asymp c_{[n^2/2]}(\mathrm{id}) \asymp \begin{cases} n^{1/2-1/p-1/q} & p \geqslant 2 \\ n^{-1/q} & p \leqslant 2 \end{cases} \tag{2}$$

$$e_{[n^2/2]}(\mathrm{id}\colon s_p^n \to s_2^n) \asymp c_{[n^2/2]}(\mathrm{id}) \asymp n^{1/2-1/p}. \tag{3}$$

Recall that for any $T \in \mathscr{L}(X, l_2^N)$

$$\left(\frac{\mathrm{vol}(TB_X)}{\mathrm{vol}(B_{l_2^N})}\right)^{1/N} \leqslant 2e_N(T)$$

(see e.g. [CS] or [P]). Since by the proposition

$$e_{[n^2/2]}(\mathrm{id}\colon l_p^n \otimes_\varepsilon l_q^n \to l_2^{n^2}) \asymp \frac{1}{e_{[n^2/2]}(\mathrm{id}\colon l_{p'}^n \otimes_\pi l_{q'}^n \to l_2^{n^2})},$$

the inverse Santalo inequality of Bourgain and Milman ([P], p. 100) yields for $\alpha = \varepsilon$ and π

$$\left(\frac{\mathrm{vol}(B_{l_p^n \otimes_\alpha l_q^n})}{\mathrm{vol}(B_{l_2^{n^2}})}\right)^{1/n^2} \asymp e_{[n^2/2]}(\mathrm{id}\colon l_p^n \otimes_\alpha l_q^n \to l_2^{n^2}),$$

hence

$$(\mathrm{vol}(B_{l_p^n \otimes_\alpha l_q^n}))^{1/n^2} \asymp \frac{1}{n} e_{[n^2/2]}(\mathrm{id}\colon l_p^n \otimes_\alpha l_q^n \to l_2^{n^2}).$$

The resulting asymptotic estimates for the volume of $B_{l_p^n \otimes_\alpha l_q^n}$ in terms of p and q are due to Schütt [S2]. Moreover, by Lemma 1 of section 4 and by section 3 this immediately gives estimates for the folume ratio of π- and ε-tensor products of l_r^n's:

$$vr(l_p^n \otimes_\alpha l_q^n) := \left(\frac{\text{vol}(B_{l_p^n \otimes_\alpha l_q^n})}{\text{vol}(D_{\max})} \right)^{1/n^2}$$
$$\asymp e_{[n^2/2]}(\text{id}: l_p^n \otimes_\alpha l_q^n \to l_2^{n^2})\, \alpha(n, n, p, q);$$

the estimates for $vr(l_p^n \otimes_\pi l_q^n)$ in terms of p and q were first given by Schütt [S2].

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